Lecture 9 - MTH 161

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BASIC DIFFERENTIATION FORMULAS

It would be tedious to always have to use definition of derivative to find derivative.

In this section we will find formulas for derivatives of constant functions, power functions, polynomials and the sine and cosine function.

1) Let $f(x) = c$ (the constant function)

Then, $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0$$

[This is expected because $f(x) = c$ is the horizontal line, and has slope 0]

So in Leibniz notation, $\boxed{\frac{d}{dx}(c) = 0}$

Power functions

a) Let $f(x) = x$

Then, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{x+h-x}{h} = \lim_{h \to 0} \frac{h}{h}$

$= \lim_{h \to 0} \frac{1}{1} = 1$

So, $\boxed{\frac{d}{dx}(x) = 1.}$

b) $f(x) = x^2$

Then $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$

$= \lim_{h \to 0} \frac{h(2x+h)}{h} = \lim_{h \to 0} 2x + h = 2x$.

So, $\boxed{\frac{d}{dx}(x^2) = 2x}$

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In general, if n is any real number, then

$$\boxed{\frac{d}{dx}(x^n) = n \cdot x^{n-1}}$$

Example

Differentiate

- $f(x) = \frac{1}{x^2}$

Step 1 We want to write it as a power function so $f(x) = \frac{1}{x^2} = x^{-2}$

Step 2
Then, $f'(x) = \frac{d}{dx}\left[f(x)\right] = \frac{d}{dx}(x^{-2}) = -2x^{-2-1} = -2x^{-3} = \frac{-2}{x^3}$

- $f(x) = \sqrt[4]{x^3}$

Step 1 We want to write it as a power function so $f(x) = (x^3)^{1/4} = x^{3/4}$

STEP 2 Then, $f'(x) = \frac{d}{dx}\left[f(x)\right] = \frac{d}{dx}(x^{3/4}) = \frac{3}{4}x^{3/4-1} = \frac{3}{4}x^{-1/3}$

Ex Find equation of the tangent line and normal line to the curve $y = x\sqrt{x}$ at pt $(1,1)$.

Soln We Know that the slope of the tangent line to a curve $y = f(x)$ at the point $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \left.\frac{dy}{dx}\right|_{x=a}$

The derivative of $f(x) = x\sqrt{x} = x \cdot x^{1/2} = x^{1+1/2} = x^{3/2}$

$$f'(x) = \frac{d}{dx}\left(x^{3/2}\right) = \frac{3}{2} x^{3/2 - 1} = \frac{3}{2} x^{1/2} = \frac{3}{2}\sqrt{x}$$

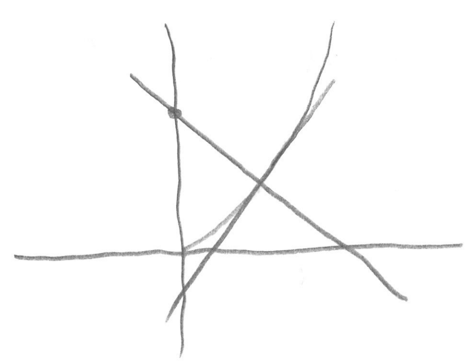
So the slope of the tangent line at $(1,1)$ is $f'(1) = \frac{3}{2}\sqrt{1} = \frac{3}{2} = m$

Therefore equation of the tangent line is $y - 1 = \frac{3}{2}(x-1) \Rightarrow \underline{y = \frac{3}{2}x - \frac{1}{2}}$

The normal line is perpendicular to the tangent line, and since slope of tangent line is $\frac{3}{2}$, the slope of the normal line is $\frac{-1}{(3/2)} = -\frac{2}{3}$

Thus the eqn of the normal line is

$$y - 1 = -\frac{2}{3}(x-1) \Rightarrow y = \frac{-2}{x} + \frac{5}{3}$$



Lec 9

- When we form new functions by adding, substracting or multiplying functions by constant, their derivatives can be calculated in term of derivatives of the old function.

- Let $g = cf(x)$

$$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{cf(x+h)-cf(x)}{h}$$

$$= \lim_{h\to 0} c\left[\frac{f(x+h)-f(x)}{h}\right] \overset{\text{Limit Law}}{=} c \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= cf'(x).$$

The constant multiple rule : If c is a constant and f is differentiable function, then $\frac{d}{dx}\left[cf(x)\right] = c\frac{d}{dx}\left[f(x)\right]$

Example

- $\frac{d}{dx}(6x^7) = 6\frac{d}{dx}(x^7) = 6.\left(7x^{7-1}\right) = 6.7x^6 = 42x^6$

- $\frac{d}{dx}\left(\frac{3}{\sqrt[3]{x}}\right) = \frac{d}{dx}\left(\frac{3}{x^{1/3}}\right) = \frac{d}{dx}\left(3x^{-1/3}\right) = 3\frac{d}{dx}\left(x^{-1/3}\right)$

$= 3.\left(-\frac{1}{3}\right)x^{-1/3-1} = -x^{-4/3}$ $\square$

- Let $F(x) = f(x) + g(x)$

Then, $F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{f(x+h)+g(x+h) - (f(x)+g(x))}{h}$

$= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}\right] = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$

$= f'(x) + g'(x).$

The sum rule If f and g are both differentiable, then

$$\boxed{\frac{d}{dx}\left(f(x)+g(x)\right) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)}$$

The sum rule can be extended to sum of any number of functions.

For instance,

$$(f+g+h)' = ((f+g)+h)' = (f+g)' + h' = f' + g' + h' \quad \boxtimes$$

By writing,

$f - g = f + (-1)g$ and applying sum rule,

$$(f-g)' = (f+(-1)g)' = f' + [(-1)g]' = f' + (-1)g' = f' - g'$$

Hence the Difference rule, If f and g are both differentiable

$$\boxed{\frac{d}{dx}\left[f(x) - g(x)\right] = \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x))}$$

- The constant rule, the sum rule, and the difference rule can be combined with the power rule to differentiate any polynomial.

Ex

$$\frac{d}{dx}\left(x^7+4x^4+13x^3-12x^2+6\right)$$

$$= \frac{d}{dx}(x^7) + 4\frac{d}{dx}(x^4) + 13\frac{d}{dx}(x^3) - 12\frac{d}{dx}(x^2) + \frac{d}{dx}(6)$$

$$= 7x^6 + 4\cdot 4x^{4-1} + 13\cdot 3\cdot x^{3-1} - 12\cdot 2x^{2-1} + 0$$

$$= 7x^6 + 16x^3 + 39x^2 - 24x$$

Ex Find the points on the curve $y = x^4 - 6x^2 + 4$ where tangent line is horizontal.

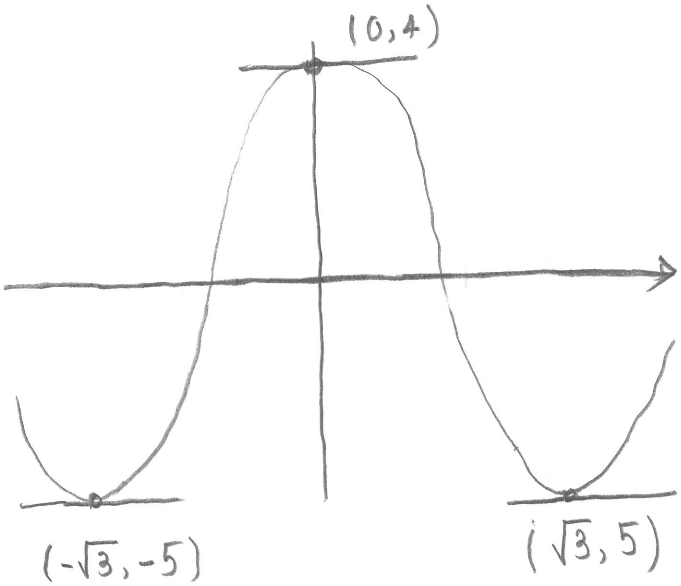
- Tangent line is horizontal means slope of tangent line is 0, and this occurs when derivative is 0.

We have $\frac{dy}{dx} = \frac{d}{dx}(x^4) - 6\frac{d}{dx}(x^2) + \frac{d}{dx}(4) = 4x^3 - 6\cdot 2x + 0 = 4x^3 - 12x$

So $\frac{dy}{dx} = 0 \Rightarrow 4x^3 - 12x = 0 \Rightarrow 4x(x^2-3) = 0 \Rightarrow x = 0$ or $x^2 - 3 = 0$

$\Rightarrow x = 0, \pm\sqrt{3}$. So the given curve has horizontal tangents when $x = 0, \sqrt{3}, -\sqrt{3}$

The corresponding pts are $(0,4)$, $(\sqrt{3}, -5)$, $(-\sqrt{3}, -5)$.



Sine and Cosine functions

Use $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ and $\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$

we can show

$$\boxed{\frac{d}{dx}(\sin(x)) = \cos x}$$

$$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

Ex Differentiate $y = 3\sin\theta + 4\cos\theta$

$$\frac{dy}{d\theta} = 3\frac{d}{d\theta}(\sin\theta) + 4\frac{d}{d\theta}(\cos\theta)$$

$$= 3\cos\theta - 4\sin\theta$$

Ex Let $f(x) = \cos x$ $\qquad f^{(4)}(x) = \cos x$

$f'(x) = -\sin x$ $\qquad f^{(5)}(x) = -\sin x$

$f''(x) = -\cos x$ $\qquad \vdots$

$f'''(x) = \sin x$

Then, $f^{(33)}(x) = f'(x) = -\sin x$

$f^{(71)}(x) = f'''(x) = \sin x$

Application of rates of change

Ex Let the position of a particle is given by the eqn

$$s = f(t) = t^3 - 6t^2 + 9t$$

where t is measured in seconds and s in meters .

a) Velocity after time t

- The velocity function is the derivative of the posn function

$$s = f(t) = t^3 - 6t^2 + 9t$$

$$v(t) = \frac{ds}{dt} = 3t^2 - 12t + 9$$

b) Velocity after 2s ? After 4s ?

$$v(2) = 3(2)^2 - 12(2) + 9 = -3 \text{ m/s}$$

$$v(4) = 3(4)^2 - 12(4) + 9 = 9 \text{ m/s}$$

c) When is the particle at rest ?

The particle is at rest when $v(t) = 0$ i.e $3t^2 - 12t + 9 = 0 \Rightarrow 3(t^2 - 4t + 3) = 0$

$\Rightarrow 3(t-1)(t-3) = 0 \quad \Rightarrow t = 1s$ or $t = 3s$.

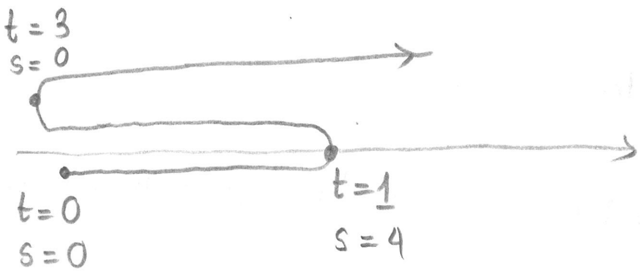
d) When is particle moving forward ?? (in the positive direction)

- The particle moves in positive dirn when $v(t) > 0$ i.e

$$3t^2 - 12t + 9 > 0 \Rightarrow 3(t-1)(t-3) > 0$$

i.e when $t < 1$ or $t > 3$, Moves backward when $1 < t < 3$

[Do an example Here]



e)

Total distance travelled by particle during the first 5 secs.

We need to calculate distances travelled during time intervals $[0,1], [1,3], [3,5]$

The distance travelled in the first sec is

$$|f(1) - f(0)| = |4 - 0| = 4m$$

When $t = 1$ to $t = 3$, the distance travelled is $|f(3) - f(1)| = |0 - 4| = 4m$.

$t = 3$ to $t = 5$, " " " is $|f(5) - f(3)| = |20 - 0| = 20\ m$

Total dist $= 28\ m$.

f) What is the acceleration function ??

The acceln is derivative of velocity function $a(t) = \frac{dv}{dt} \left(= \frac{d^2s}{dt^2}\right) = 6t - 12$